

Mixing Flows with Homogeneous Spectrum of Multiplicity Two

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1 Introduction

We consider measure-preserving flows on a standard probability space (X, μ) . A flow T_t is mixing if for all measurable sets $A, B \subset X$

$$\mu(T_t A \cap B) \rightarrow \mu(A)\mu(B), \quad t \rightarrow \infty.$$

Our aim is to obtain a mixing flow with homogeneous spectrum of multiplicity 2. This problem is connected with Rokhlin's question on the realization of non-simple homogeneous spectrum for an ergodic dynamical system. Non-mixing ergodic transformations with such spectrum appeared in [2],[8]: for a generic transformation T the product $T \otimes T$ had homogeneous spectrum of multiplicity 2. There are several generalizations of this result, see for example [3], [4],[5], [6], [7]. Recently the realization of homogeneous spectrum of arbitrary multiplicity was made for mixing \mathbf{Z} -actions [11]. However in case of mixing flows (\mathbf{R} -actions) the problem is open.

In the present note we adapt the method of [9] to realize multiplicity 2 for mixing flows. First we get non-mixing flows which could be close to mixing ones and possess the desired properties. Let Θ denote the orthogonal projection onto the constant functions in $L_2(X, \mu)$.

Theorem 1. *For given ε , $0 < \varepsilon < 1$, let T_t be an ergodic flow with simple spectrum such that its weak closure contains operators*

$$(1 - \varepsilon)\Theta + \frac{\varepsilon}{a} \int_0^a T_s ds$$

for all $a > 0$.

Then

- (1) *the spectral measure σ of T_t and the convolution $\sigma * \sigma$ are disjoint*

- (2) the symmetric tensor square $T_t \odot T_t$ has simple spectrum;
- (3) $T_t \otimes T_t$ has homogeneous spectrum of multiplicity 2.

Then via some approximation procedure we find a mixing flow T_t such that $T_t \odot T_t$ has simple spectrum. This implies the following assertion.

Theorem 2. *There is a mixing flow T_t such that $T_t \otimes T_t$ has homogeneous spectrum of multiplicity 2.*

2 Auxiliary non-mixing flows. Mixing limit flows

Cutting-and-staking rank-one flow construction is determined by a parameter h_1 , a cut sequence r_j and a spacer sequence \bar{s}_j ,

$$\bar{s}_j = (s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j)),$$

where $s_j(i) \in \mathbf{R}^+$.

Constructions. We consider a special class F_ε , $0 < \varepsilon < 1$, of rank-one flows, setting (ε is fixed)

$$r_j := j,$$

$$s_j(i) := \frac{i}{\sqrt{j}}$$

as $1 \leq i \leq (1 - \varepsilon)j$,

$$s_j(i) := \frac{i - (1 - \varepsilon)j}{\sqrt{j^3}}$$

as $(1 - \varepsilon)j < i \leq j$. Starting from h_1 let's define a sequence h_j :

$$h_{j+1} = h_j r_j + \sum_{i=1}^{r_j} s_j(i).$$

Standard "rank-one" calculations show that

$$T_{h_j} \rightarrow_w (1 - \varepsilon)\Theta + \varepsilon I.$$

So such a flow is $(1 - \varepsilon)$ -mixing in sense of [10]. In fact the weak closure of this flow contains all limit operators mentioned in Theorem 1.

Lemma. *For any $a > 0$ the weak closure of a flow $T_t \in F_\varepsilon$ contains an operator in the form*

$$(1 - \varepsilon)\Theta + \frac{\varepsilon}{a} \int_0^a T_s ds.$$

For an automorphism T with simple spectrum the existence of power limits in the form

$$\frac{1}{q}(I + T + T^2 + \dots + T^{q-2} + \Theta)$$

implies that $T \otimes T$ has homogeneous spectrum of multiplicity 2 [9]. Theorem 1 is an analog of this result, its proof is similar to the proof of Theorem 4.2 from [9].

To prove Theorem 2 we consider a sequence $\varepsilon_j \rightarrow 0$ which tends to 0 extremely slowly. We consider a rank-one flow T_t setting by a fixed h_1

$$r_j := j,$$

$$s_j(i) := \frac{i}{\sqrt{j}}$$

as $1 \leq i \leq (1 - \varepsilon_j)j$,

$$s_j(i) := \frac{i - (1 - \varepsilon_j)j}{\sqrt{j^3}}$$

as $(1 - \varepsilon_j)j < i \leq j$.

If $\varepsilon_j \rightarrow 0$ remains constant for a very long time, then the corresponding construction is well approximated by flows from classes F_ε , and this provides simple spectrum of $T_t \odot T_t$. The latter implies homogeneous spectrum of multiplicity 2 for $T_t \otimes T_t$. Because of $\varepsilon_j \rightarrow 0$ our flow T_t becomes almost staircase rank-one construction. In a view of [1] (mod some modification) T_t is mixing.

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